Lecture 8

Friday, October 18, 2019 5:38 AM

- Finish Thm 4 from Lecture 7 notes.

<u>Uniform convergence</u>.

<u>Def</u>. If f, $\{f_n\}_{n=1}^{\infty}$ are functions $f, f_n : (X, d) \to (\Omega, \rho)$ then $\{f_n\}$ <u>converges uniformly</u> to f, $f_n \to f$ unif., if $\forall \varepsilon > 0$ $\exists N$ s.t. $\rho(f_n(x), f(x)) < \varepsilon$ when $n \geq N$.

<u>Thm 1</u>. If $f_n : X \to \Omega$ are cont. for all $n \in \mathbb{N}$ and $f_n \to f$ unif. $\Rightarrow$ $f : X \to \Omega$ is continuous

<u>Pf</u>. Pick $a \in X$ and $\varepsilon > 0$. $\exists N$ s.t. $\rho(f_n(x), f(x)) < \varepsilon/3$, $\forall n \geq N$. f_N cont. $\Rightarrow$ $\exists \delta > 0$ s.t. $\rho(f_N(x), f_N(a)) < \varepsilon/3$, $d(x,a) < \delta$. Thus, if $d(x,a) \Rightarrow \rho(f(x), f(a)) \overset{\Delta\text{-ineq.}}{\leq} \rho(f(x), f_N(x)) + \rho(f_N(x), f(a))$ $\overset{\Delta\text{-ineq.}}{\leq} \varepsilon/3 + \rho(f_N(x), f_N(a)) + \rho(f_N(a), f(a)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.
$\Rightarrow f$ is cont. ▨

<u>Thm 2</u>. (Weierstrass Majorant (M) test). Let $u_n : X \to \mathbb{C}$ be functions s.t. $|u_n(x)| \leq M_n$ $\forall n \geq N$ (some N) and $\forall x \in X$. If $\sum_{n=1}^{\infty} M_n < \infty \Rightarrow \sum_{n=1}^{\infty} u_n(x)$ converges unif. to some function $f : X \to \mathbb{C}$.

<u>Pf</u>. We first define f. Let $f_n(x) := \sum_{k=1}^{n} u_k(x)$. For fixed $x_0 \in X$, we have $|f_n(x_0) - f_m(x_0)| \leq \sum_{k=m}^{n} |u_k(x_0)| \leq \sum_{k=m}^{n} M_k$. Since $\sum_{k=1}^{\infty} M_k < \infty$, $\{f_n(x_0)\}_{n=1}^{\infty}$ is Cauchy seq. in $\mathbb{C}$. Since $\mathbb{C}$ complete $\exists f(x_0) \in \mathbb{C}$ s.t. $f_n(x_0) \to f(x_0)$ in $\mathbb{C}$. This defines $f : X \to \mathbb{C}$. Claim

s.t. $f_n(x_0) \to f(x_0)$ in $\mathbb{C}$. This defines $f: X \to \mathbb{C}$. Claim that $f_n \to f$ unif. Pick $\varepsilon > 0$. $\exists N$ s.t. $\sum_{n=N+1}^{\infty} M_n < \varepsilon$. Thus, for any $x \in X$, $|f_n(x) - f(x)| \leq \sum_{k=n+1}^{\infty} |u_k(x)| \leq \sum_{k=n+1}^{\infty} M_n < \varepsilon$ if $n \geq N$; i.e, $f_n \to f$ unif. $\boxtimes$

Remarks: ① A special case occurs if $X = \{x_0\}$ (one point). Then the u_n's are just complex numbers a_n. Thus, we get a seq. $\{a_n\}_{n=1}^{\infty}$ of complex numbers. The assumption in Thm 2 is just that $\sum_{n=1}^{\infty} |a_n| < \infty$; we say $\sum_{n=1}^{\infty} a_n$ converges absolutely. The conclusion is that $\exists w \in \mathbb{C}$ s.t. $\sum_{n=1}^{\infty} a_n$ converges to w, $\sum_{n=1}^{\infty} a_n = w$, i.e.

$$\left| \sum_{k=1}^{n} a_k - w \right| \to 0 \text{ as } n \to \infty.$$

② If the functions $u_n : X \to \mathbb{C}$ are all cont., then the partial sums $f_n = \sum_{k=1}^{n} u_n$ are cont. Since $f_n \to f$ unif., by Thm 1, the limit $f = \sum_{n=1}^{\infty} u_n : X \to \mathbb{C}$ is continuous.